MATH 1A - MOCK MIDTERM 3 - SOLUTIONS

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1. *(25 points)* Sketch a graph of the function $f(x) = e^{\frac{-x^2}{2}}$.

1) Domain: $\mathbb{R}$ (all real numbers)

2) Intercepts: $y-$intercept $f(0) = 1$, no $x-$intercept as $f(x)$ is always positive.

3) Symmetry: $f(-x) = f(x)$, so f is even

4) Asymptotes: No vertical asymptotes, as f is defined everywhere.

   However:

$$\lim_{x \to \pm\infty} f(x) = e^{-\infty} = 0$$

   So $y = 0$ is a horizontal asymptote at $\pm\infty$

5) Intervals of increase/decrease and Local max/min:

$$f'(x) = e^{-\frac{x^2}{2}} \left(-\frac{1}{2}\right)(2x) = e^{-\frac{x^2}{2}}(-x)$$

   So f is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.
   $f(0) = 1$ is a local maximum.

6) Concavity

$$f''(x) = e^{-\frac{x^2}{2}}(-x)(-x) + e^{-\frac{x^2}{2}}(-1) = e^{-\frac{x^2}{2}}\left(x^2 - 1\right)$$

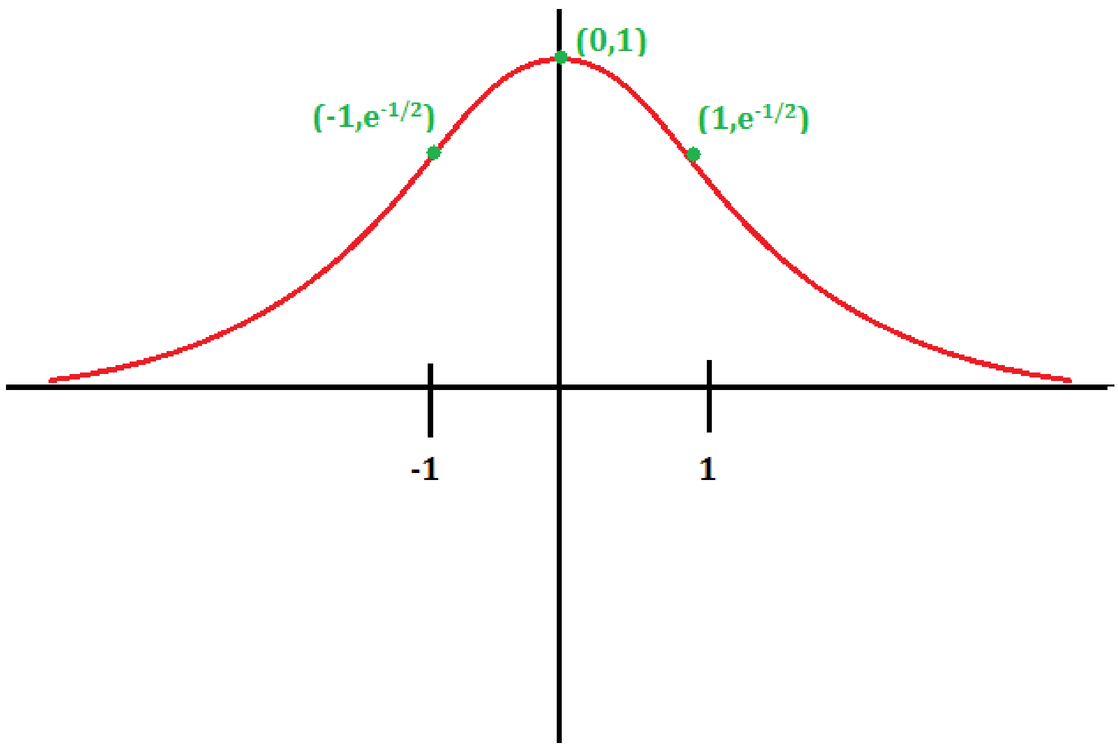
   Then $f''(x) = 0 \Leftrightarrow x^2 - 1 = 0 \Leftrightarrow x = \pm 1$.

Date: Friday, July 22nd, 2011.

So f is concave up on $(-\infty, -1)$, concave down on $(-1, 1)$ and concave up on $(1, \infty)$ (draw a sign table if you want).
$(\pm 1, e^{-\frac{1}{2}})$ are inflection points.

7) Graph

1A/Math 1A Summer/Exams/Normal.png



2. *(10 points)* Use linear approximations (or differentials) to find an approximate value of $(3.01)^3$

Linear approximation:

Let $f(x) = x^3$, $a = 3$. Then $f(3) = 27$ and $f'(3) = 3(3)^2 = 27$, so

$$L(x) = f(a) + f'(a)(x - a) = 27 + 27(x - 3)$$

Now:

$$(3.01)^3 = f(3.01) \approx L(3.01) = 27+27(3.01-3) = 27+27(0.01) = 27.27$$

Differentials:

Let $f(x) = x^3$, $x = 3$, $dx = 3.01 - 3 = 0.01$. Then:

$$dy = f'(x)dx = 3(3)^2(0.01) = 27(0.01) = 0.27$$

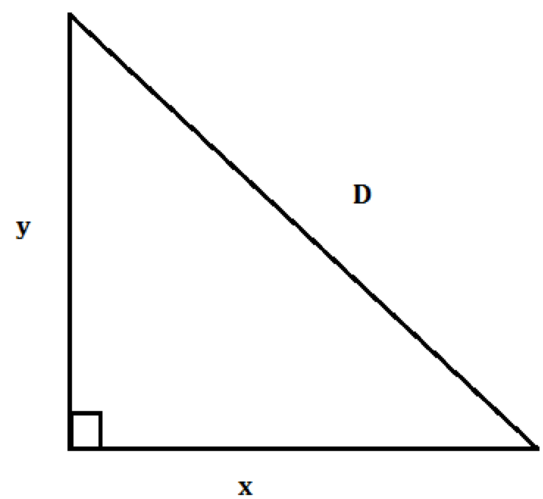
And

$$(3.01)^3 \approx f(3) + dy = 27 + 0.27 = 27.27$$

3. *(15 points)* Two people start moving from the same point. One person travels north at a speed of 3 mph and the other person travels east at a speed of 4 mph. At what rate is the distance between the two people changing after 2 hours?

1) Picture

1A/Math 1A Summer/Exams/Mock3triangle.png



2) WTF: $\frac{dD}{dt}$ after $2h$

3) $D^2 = x^2 + y^2$

4) $2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

5) Now $\frac{dx}{dt} = 4$, $\frac{dy}{dt} = 3$, and:

$x =$ Speed $\times$ Time $4 \times 2 = 8$
$y =$ Speed $\times$ Time $3 \times 2 = 6$.

If you draw another triangle with the values of $x = 8$ and $y = 6$ plugged in, you notice that it's a $6 - 8 - 10$ triangle, and so $D = 10$.

Finally, we get:

6)

$$2(10)\frac{dD}{dt} = 2(8)(4) + 2(6)(3)$$

$$20\frac{dD}{dt} = 64 + 36 = 100$$

$$\frac{dD}{dt} = \frac{100}{20} = 5mph$$

4. *(10 points)* Evaluate the following limits:

(a)

$$\lim_{x\to 0} \frac{x+\sin(x)}{1+\cos(x)} = \frac{0}{2} = 0$$

Trick question! Before using, l'Hopital's rule, always check whether your limit is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$)

(b)

$$\lim_{x\to -\infty} xe^x = \lim_{x\to -\infty} \frac{x}{e^{-x}} \stackrel{H}{=} \lim_{x\to -\infty} \frac{1}{-e^{-x}} = \frac{1}{-\infty} = 0$$

(c) $\lim_{x\to 0} x^x$

1) $y = x^x$
2) $\ln(y) = x\ln(x)$
3)

$$\lim_{x\to 0} \ln(y) = \lim_{x\to 0} x\ln(x) = \lim_{x\to 0} \frac{\ln(x)}{\frac{1}{x}} \stackrel{H}{=} \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0} -x = 0$$

4)

$$\lim_{x\to 0} x^x = e^0 = 1$$

5. *(10 points)* Find the absolute maximum and minimum of $f(x) = x^3 - 3x$ on $[0, 2]$

1) Endpoints: $f(0) = 0$, $f(2) = 8 - 6 = 2$

2) Critical numbers:

$$f'(x) = 3x^2 - 2 = 3(x^2 - 1)$$

So $f'(x) = 0$ if $x^2 - 1 = 0$, so $x = \pm 1$. However, -1 is not in $[0, 2]$ so ignore it, hence the only critical number is 1, and $f(1) = 1 - 3 = -2$

3) Compare:

The absolute minimum of f is $f(1) = -2$ and the absolute maximum of f is $f(2) = 2$.

6. *(10 points)* Suppose f is an odd function and is differentiable everywhere. Let b be given. Show that there is a number c in $(-b, b)$ such that $f'(c) = \frac{f(b)}{b}$

Hint: Let $a = -b$

By the Mean Value theorem, for some c, we have:

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

However, now let $a = -b$, so we get:

$$\frac{f(b) - f(-b)}{b - (-b)} = f'(c)$$

But f is odd, so $f(-b) = -f(b)$:

$$\frac{f(b) - (-f(b))}{b - (-b)} = f'(c)$$

$$\frac{2f(b)}{2b} = f'(c)$$

$$\frac{f(b)}{b} = f'(c)$$

Which is what we wanted!

7. *(20 points)* Find the point on the line $x + y = 1$ that is closest to the point $(-3, 1)$.

1) Picture: Just draw the line $y = 1 - x$, the point $(-3, 1)$ and a point (x, y) on the line. Your picture does not have to be accurate, just to give you an idea of what's going on!

2) Want to minimize $D = \sqrt{(x+3)^2 + (y-1)^2}$

However, $y = 1 - x$, so:

$$D = \sqrt{(x+3)^2 + (y-1)^2} = \sqrt{(x+3)^2 + (1-x-1)^2} = \sqrt{(x+3)^2 + x^2}$$

Bu, in order to get rid of the square root, instead of maximizing D, let's maximize f, where:

$$f(x) = (D(x))^2 = (x+3)^2 + x^2$$

3) Constraint: None (the point (x, y) moves freely on the line)

4)
$$f'(x) = 2(x+3) + 2x = 4x + 6$$

So $f'(x) = 0$ when $4x + 6 = 0$, i.e. when $x = \frac{-6}{4} = -\frac{3}{2}$.

In particular, then we get $y = 1 - x = 1 + \frac{3}{2} = \frac{5}{2}$.

So our answer is $\boxed{(x, y) = (-\frac{3}{2}, \frac{5}{2})}$

Bonus 1 *(5 points)* Use l'Hopital's rule to show:

$$\lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x)$$

Note: Careful! You're differentiating with respect to h here, not x !

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} &\overset{H}{=} \lim_{h\to 0} \frac{f'(x+h) - 0 - f'(x-h)}{2h} \\
&\overset{H}{=} \frac{f''(x+h) + f''(x-h)}{2} \\
&= \frac{f''(x) + f''(x)}{2} \\
&= f''(x)
\end{aligned}$$

Bonus 2 *(5 points)* (Courtesy Adi Adiredja)

Omar didn't get *the* girl. Sad and frustrated he decided to run around the track at the gym. When he got to the track, one girl who was about to start running captured his attention. Omar hurried so they could start running together. They started at the same time, but then she started running faster. Omar sped up, but then she passed him again. Next thing he knew, the two of them were racing. At some point she noticed the finish line, and yelled out,"I dont go for losers!" Omar ran as fast as he could and the race ended in a tie. After the race she confidently came up to him and said, "I dont *mean* to be rude, but I only *value* smart guys. If you can prove the next thing I say, I'll go on a date with you."
She then said, "We started the race at the same time, and the race ended in a tie, I claim that at some point during the race we were running at the same speed." She smiled at Omar and said, "Are you smart enough?" Use Calculus to help Omar!

Hint: Let $g(t)$ and $h(t)$ be the position functions of the two runners, and consider $f(t) = g(t) - h(t)$

Consider $g(t)$ and $h(t)$, the position functions of the runners. Define $\boxed{f(t) = g(t) - h(t)}$. Then $f(0) = g(0) - h(0) = 0$ (since the runners started at the same place), and $f(T) = g(T) - h(T) = 0$, where T is the ending time (we know $g(T) = h(T)$ because the race ended in a tie). But then $f(0) = f(T)$, so by Rolle's theorem, $\boxed{f'(c) = 0}$ for some c in $(0, T)$. But $f'(c) = g'(c) - h'(c)$, so $g'(c) - h'(c) = 0$, so $g'(c) = h'(c)$, but this says precisely that at some point in time (namely at c), the two runners had the same speed!